

# Minimal counterexamples and discharging method

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## Abstract

Recently, the author found that there is a common mistake in some papers by using minimal counterexample and discharging method. We first discuss how the mistake is generated, and give a method to fix the mistake. As an illustration, we consider total coloring of planar or toroidal graphs, and show that: if  $G$  is a planar or toroidal graph with maximum degree at most  $\kappa - 1$ , where  $\kappa \geq 11$ , then the total chromatic number is at most  $\kappa$ .

## 1 Introduction

A graph property  $\mathcal{P}$  is *deletion-closed* if  $\mathcal{P}$  is closed under taking subgraphs. We denote the minimum degree and maximum degree of a graph  $G$  by  $\delta(G)$  and  $\Delta(G)$ , respectively. We denote  $\xi(G)$  a parameter of  $G$ , such as total chromatic number, list chromatic index, list total chromatic number, and so on. We denote  $\zeta(G)$  a function of  $\Delta(G)$ , and denote  $\lambda_1, \lambda_2, \kappa$  positive integers. Most of the results regarding planar graphs or toroidal graphs were proved by taking a minimal counterexample and using discharging method. Recently, the author found that there are many papers investigated results in the following form.

(\*) Let  $G$  be a planar or toroidal graph with deletion-closed property  $\mathcal{P}$ . If  $\Delta(G) \geq \lambda_1$ , then  $\xi(G) \leq \zeta(G)$ .  $\square$

In the proof, they wrote “Let  $G$  be a minimal counterexample. By the minimality of  $G$ , we have that  $\xi(G - e) \leq \zeta(G)$ .” But something has been ignored, thus the argument is wrong because we cannot guarantee  $\Delta(G - e) \geq \lambda_1$ , that is, the condition  $\Delta(G) \geq \lambda_1$  is not deletion-closed, so we cannot use the minimality of  $G$ . Therefore, some researchers changed to prove the corresponding results in the following form.

( $\diamond$ ) Let  $G$  be a planar or toroidal graph with deletion-closed property  $\mathcal{P}$ . If the maximum degree is at most  $\lambda_2$ , where  $\lambda_2 \geq \lambda_1$ , then  $\xi(G) \leq \zeta(G)$ .  $\square$

Hence, most of proofs about planar graphs can be fixed by changing the statement to the above form ( $\diamond$ ). But for the toroidal graphs, most of the proofs cannot be fixed even you adopt the above form ( $\diamond$ ). In the proof, to derive a contradiction, after the discharging process, we need to show that at least one element (vertex/face) has positive final charge. The common doing is to show the final charge of  $\lambda_2$ -vertex is positive, but maybe  $\Delta(G) < \lambda_2$  and there is no  $\lambda_2$ -vertex.

Until now, the author found the results in [7, 13, 16, 19, 25, 26, 30, 32] and the corollaries in [6, 14, 15, 22, 27] are wrong. To the author’s knowledge, the earliest paper having this problem is Zhao’s paper [32] on total coloring, thus we only consider the total coloring problem.

A *total coloring* of a graph  $G$  is an assignment of colors to the vertices and edges of  $G$  such that every pair of adjacent/incident elements receive distinct colors. The *total chromatic number* of a graph  $G$ , denoted by  $\chi''(G)$ , is the minimum number of colors in a total coloring of  $G$ . It is obvious that the total chromatic number of a graph  $G$  has a trivial lower bound  $\Delta(G) + 1$ . For the upper bound, Behzad [1] raised the following well-known Total Coloring Conjecture (TCC):

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**Total Coloring Conjecture.** Every graph with maximum degree  $\Delta$  admits a total coloring with at most  $\Delta + 2$  colors.

The conjecture was verified in the case  $\Delta = 3$  by Rosenfeld [18] and Vijayaditya [23] independently and also by Yap [31]. It was confirmed in the case  $\Delta \in \{4, 5\}$  by Kostochka [10, 11], in fact the proof holds for multigraphs. Regarding planar graphs, the conjecture was verified in the case  $\Delta \geq 9$  by Borodin [2] and in the case  $\Delta = 7$  by Sanders and Zhao [20]; the case  $\Delta = 8$  was a consequence of Vizing's theorem about planar graphs [24] and four coloring theorem (for more details, see Jensen and Toft [8]). Thus, the only remaining case for planar graphs is that of maximum degree six. Note that best known upper bound on the total chromatic number of planar graph with maximum degree 6 is 9 [2].

For planar graphs with large maximum degree, the total chromatic number can be obtained. Precisely, Borodin [2] showed that if  $\Delta \geq 14$  then  $\chi''(G) = \Delta(G) + 1$ . Borodin, Kostochka and Woodall improved the result to the case  $\Delta \geq 12$  [3] and  $\Delta = 11$  [4]. Recently, Wang [29] further improved the result for  $\Delta = 10$ , and Kowalik et al. [12] improved the result for  $\Delta = 9$ .

In section 2, we give some structural results which are very helpful in the proof of total coloring problem. In section 3, we give an illustration how to prove the statement in the revised form ( $\diamond$ ).

## 2 Total $\kappa$ -coloring

A  $\kappa$ -deletion-minimal graph with respect to total coloring, is a graph with maximum degree at most  $\kappa - 1$  such that its total chromatic number is greater than  $\kappa$ , but the total chromatic number of every proper subgraph is at most  $\kappa$ . In this section, we give many structural results on  $\kappa$ -deletion-minimal graph  $G$ , most of which can be obtained by trivially extending the corresponding proofs in other papers. Note that some of the results in this section may be not used in section 3, and we just collect as many results as possible. All the solid black dots are only incident with the edges depicted in the figures.

Usually, we first give a partial total coloring of  $G$ , and then we extend the coloring to  $G$  in the proof. Since an uncolored vertex with degree at most  $\lfloor \frac{\kappa-1}{2} \rfloor$  forbids at most  $2 \deg \leq 2 \lfloor \frac{\kappa-1}{2} \rfloor \leq \kappa - 1$  colors, so we always have at least one available color for the vertex, thus we will not care about the coloring of the vertices with degree at most  $\lfloor \frac{\kappa-1}{2} \rfloor$ .

A vertex of degree  $\tau$ , at most  $\tau$  and at least  $\tau$  are called a  $\tau$ -vertex,  $\tau^-$ -vertex and  $\tau^+$ -vertex, respectively. Let  $[\kappa]$  denote the set  $\{1, 2, \dots, \kappa\}$ . We denote  $\mathcal{U}(w)$  the set of colors which are assigned to the vertex  $w$  and edges incident with  $w$ .

**Lemma 1.** The graph  $G$  is 2-connected.

**Lemma 2** (Wang [28]). If  $u$  and  $v$  are two adjacent vertices with  $\deg_G(v) \leq \lfloor \frac{\kappa-1}{2} \rfloor$ , then  $\deg_G(u) + \deg_G(v) \geq \kappa + 1$ .

**Lemma 3.** The minimum degree is at least  $\kappa - \Delta(G) + 1$ .

**Lemma 4.** If  $\kappa \geq 5$ , then the subgraph induced by the edges incident with 2-vertices is a forest.

**Proof.** Firstly, by Lemma 2, the set of 2-vertices is independent and the edge induced subgraph is bipartite. Suppose that it contains a cycle  $C$ . By the minimality of  $G$ , the graph  $G - E(C)$  admits a total coloring  $\varphi$  with at most  $\kappa$  colors. We can extend  $\varphi$  to  $G$  by using the known result that every even cycle is 2-edge-choosable, which leads to a contradiction.  $\square$

**Lemma 5.** Let  $u$  and  $v$  be two adjacent vertices with  $\deg_G(v) \leq \lfloor \frac{\kappa-1}{2} \rfloor$  and  $\deg_G(u) + \deg_G(v) \leq \kappa + 1$ . If  $uv$  is contained in a triangle  $uvw$ , then  $\deg_G(w) = \kappa - 1$ .

**Proof.** By contradiction, suppose that  $\deg_G(w) \leq \kappa - 2$ . By the minimality of  $G$ , the graph  $G - uv$  admits a total coloring with at most  $\kappa$  colors. Now, we erase the color on the vertex  $v$ , and denote the resulting coloring by  $\varphi$ . If  $\{1, 2, \dots, \kappa\}$  is not the union of  $\mathcal{U}_\varphi(u)$  and  $\mathcal{U}_\varphi(v)$ , then we can extend the coloring  $\varphi$  to  $uv$ . Hence, the set  $\{1, 2, \dots, \kappa\}$  is the union of  $\mathcal{U}_\varphi(u)$  and  $\mathcal{U}_\varphi(v)$ ; in fact, it is the disjoint union of  $\mathcal{U}_\varphi(u)$  and  $\mathcal{U}_\varphi(v)$  since  $|\mathcal{U}_\varphi(u)| + |\mathcal{U}_\varphi(v)| = \kappa$ . Note that  $\varphi(uv) \notin \mathcal{U}_\varphi(u)$ . Let  $\phi$  be the coloring from  $\varphi$  by assigning the color  $\varphi(uv)$  to  $uv$  and erasing the color on  $wv$ . Similarly, we can prove that  $\{1, 2, \dots, \kappa\}$  is the union (not necessarily disjoint union) of  $\mathcal{U}_\phi(w)$  and  $\mathcal{U}_\phi(v)$ . Therefore, we have  $\mathcal{U}_\varphi(u) \subseteq \mathcal{U}_\phi(w) \subset \mathcal{U}_\phi(v)$ . Since  $\deg_G(w) \leq \kappa - 2$ , it follows that there exists a color  $\alpha \notin \mathcal{U}_\varphi(u) \cup \mathcal{U}_\varphi(v)$ . Note that  $\varphi(uw) \notin \mathcal{U}_\varphi(v)$ . We extend  $\varphi$  by assigning  $\alpha$  to  $uw$  and assigning  $\varphi(uw)$  to  $uv$ .  $\square$

**Lemma 6.** If  $\kappa = 2\tau$ , then  $(\tau, \tau)$ -edge is not contained in a triangle.

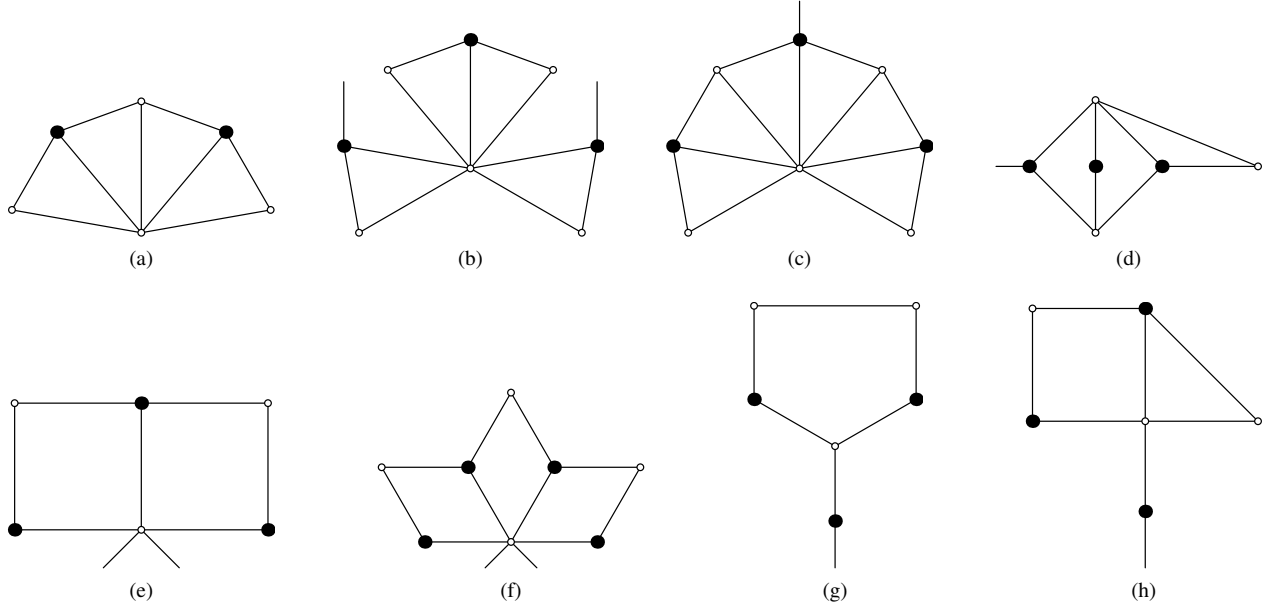


Fig. 1: Reducible configurations

**Proof.** (See also Lemma 2 (vi) in [12]) By contradiction, suppose that a  $(\tau, \tau)$ -edge  $uv$  is contained in a triangle  $uvw$ . By the minimality of  $G$ , the graph  $G - uv$  admits a total coloring with at most  $\kappa$  colors. Now, we erase the color on the vertex  $v$ , and denote the resulting coloring by  $\varphi$ . Let  $A_\varphi(uv)$  be the set of available colors for the edge  $uv$ , and  $A_\varphi(v)$  the set of available colors for the vertex  $v$ . If there exist  $\alpha_1 \in A_\varphi(uv)$  and  $\alpha_2 \in A_\varphi(v)$  such that  $\alpha_1 \neq \alpha_2$ , then we can extend  $\varphi$  by assigning  $\alpha_1$  to  $uv$  and  $\alpha_2$  to  $v$ . So we may assume that  $A_\varphi(uv) = A_\varphi(v) = \{\alpha\}$ . Hence, we have that  $|\mathcal{U}_\varphi(u) \cup \mathcal{U}_\varphi(v)| = \kappa - 1$  and  $\mathcal{U}_\varphi(u) \cap \mathcal{U}_\varphi(v) = \emptyset$ . Exchanging the colors on  $uw$  and  $vw$ , and assigning  $\alpha$  to  $uv$  and  $\varphi(wv)$  to  $v$ .  $\square$

**Lemma 7.** Let  $w$  be a vertex with  $\{w_1, w_2, w_3\} \subseteq N_G(w)$  and  $w_1w_2 \in E(G)$ . If  $\deg(w) + \deg(w_1) = \kappa + 1$  and  $\deg(w_1) \leq \lfloor \frac{\kappa-1}{2} \rfloor$ , then  $\deg(w) + \deg(w_3) \geq \kappa + 2$ .

**Proof.** Suppose that  $\deg(w) + \deg(w_3) \leq \kappa + 1$ , which implies that  $\deg(w_3) \leq \lfloor \frac{\kappa-1}{2} \rfloor$ . The graph  $G - ww_1$  admits a total coloring with at most  $\kappa$  colors. Now, we erase the colors on the vertices  $w_1$  and  $w_3$ , and denote the resulting coloring by  $\varphi$ . Notice that  $\{1, 2, \dots, \kappa\}$  is the disjoint union of  $\mathcal{U}_\varphi(w)$  and  $\mathcal{U}_\varphi(w_1)$ . Notice also that  $\mathcal{U}_\varphi(w) \cup \mathcal{U}_\varphi(w_3) = \{1, 2, \dots, \kappa\}$ ; otherwise, reassigning  $\varphi(ww_3)$  to  $ww_1$  and assigning a color in  $[\kappa] \setminus (\mathcal{U}_\varphi(w) \cup \mathcal{U}_\varphi(w_3))$  to  $w_3$ . Note that  $\varphi(ww_2) \notin \mathcal{U}_\varphi(w_3)$ . Now, exchanging the colors on  $ww_2$  and  $w_1w_2$ , and reassigning  $\varphi(ww_2)$  to  $ww_3$  and  $\varphi(ww_3)$  to  $w_1$ .  $\square$

**Lemma 8.** Let  $ww_2$  be contained in two triangles  $ww_1w_2$  and  $ww_2w_3$ . If  $\deg(w) + \deg(w_2) = \kappa + 1$  and  $\deg(w_2) \leq \lfloor \frac{\kappa-1}{2} \rfloor$ , then all the other neighbors of  $w$  have degree at least  $\deg(w_2) + 2$  or  $\lfloor \frac{\kappa+1}{2} \rfloor$ .

**Proof.** Suppose that  $w_4$  is a vertex with degree at most  $\lfloor \frac{\kappa-1}{2} \rfloor$ . The graph  $G - ww_2$  admits a total coloring with at most  $\kappa$  colors. Now, we erase the colors on the vertices  $w_2$  and  $w_4$ , and denote the resulting coloring by  $\varphi$ . Notice that  $\{1, 2, \dots, \kappa\}$  is the disjoint union of  $\mathcal{U}_\varphi(w)$  and  $\mathcal{U}_\varphi(w_2)$ . Notice also that  $\mathcal{U}_\varphi(w) \cup \mathcal{U}_\varphi(w_4) = \{1, 2, \dots, \kappa\}$ ; otherwise, reassigning  $\varphi(ww_4)$  to  $ww_2$  and assigning a color in  $[\kappa] \setminus (\mathcal{U}_\varphi(w) \cup \mathcal{U}_\varphi(w_4))$  to  $w_4$ . This implies that  $\mathcal{U}_\varphi(w_2) \subset \mathcal{U}_\varphi(w_4)$ . Now, exchanging the colors on  $ww_1$  and  $w_1w_2$ , and additionally exchanging the colors on  $ww_3$  and  $w_3w_2$ , we obtain another partial total coloring  $\sigma$ . Similarly, we have that  $\mathcal{U}_\sigma(w_2) \subset \mathcal{U}_\sigma(w_4) = \mathcal{U}_\varphi(w_4)$ , which implies that  $\varphi(ww_1), \varphi(ww_3) \subseteq \mathcal{U}_\varphi(w_4)$ . Hence, we have that  $\deg(w_4) \geq |\mathcal{U}_\varphi(w_2)| + |\{\varphi(ww_1), \varphi(ww_3), \varphi(ww_4)\}| = \deg(w_2) + 2$ .  $\square$

**Lemma 9.** Let  $ww_2$  be contained in two triangles  $ww_1w_2$  and  $ww_2w_3$ . If  $\kappa \geq 7$  and  $w_1$  is a 2-vertex, then  $\deg(w_3) \geq 4$ .

**Proof.** (See also Lemma 3 (iv) in [12]). By contradiction, suppose that  $w_3$  is a  $3^-$ -vertex. By the minimality of  $G$ , the graph  $G - ww_1$  has a total coloring with at most  $\kappa$  colors. Now, we erase the colors on the vertices  $w_1$  and  $w_3$ , and

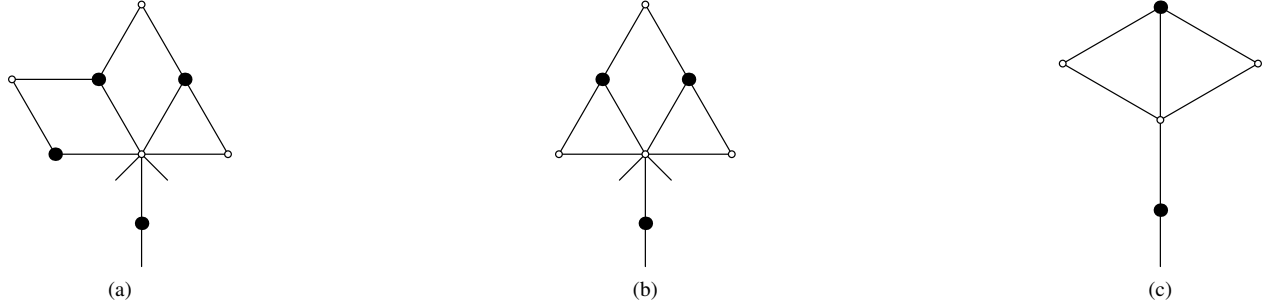


Fig. 2: Reducible configurations

denote the resulting coloring by  $\varphi$ . Thus,  $[\kappa]$  is the disjoint union of  $\varphi(w_1w_2)$  and  $\mathcal{U}_\varphi(w)$ ; otherwise, we can assign an available color to  $ww_1$ . If  $\varphi(w_1w_2) \notin \mathcal{U}_\varphi(w_3)$ , then we recolor  $ww_1$  and  $ww_3$  with  $\varphi(w_1w_2)$  and  $\varphi(w_1w_3)$ , respectively. Thus, we have  $\varphi(w_1w_2) \in \mathcal{U}_\varphi(w_3)$ , but  $\varphi(w_1w_2) \notin \{\varphi(w_2w_3), \varphi(w_1w_3)\}$ . Now, we recolor  $ww_1, w_1w_2, ww_2$  and  $ww_3$  with  $\varphi(w_1w_2), \varphi(w_1w_3), \varphi(w_1w_2)$  and  $\varphi(w_1w_3)$ , respectively.  $\square$

**Lemma 10.** If  $\kappa \geq 7$  and  $uvw$  is a  $(2, \Delta, \Delta)$ -triangle with  $\deg(u) = 2$ , then  $v$  is not contained in another  $(\Delta, 3, *)$ -triangle.

**Proof.** See [12, Lemma 3 (v)].  $\square$

**Lemma 11.** If  $\kappa \geq 7$ , then the graph  $G$  contains no configuration in Fig. 1a.

**Proof.** See [12, Lemma 6].  $\square$

**Lemma 12.** If  $\kappa \geq 7$ , then the graph  $G$  contains no configuration in Fig. 1b.

**Proof.** See [12, Lemma 4].  $\square$

**Lemma 13.** If  $\kappa \geq 9$ , then the graph  $G$  contains no configuration in Fig. 1c.

**Proof.** See [12, Lemma 7].  $\square$

**Lemma 14.** If  $\kappa \geq 7$ , then the graph  $G$  contains no configuration in Fig. 1d.

**Proof.** See [4] or [12, Lemma 3 (vi)].  $\square$

**Lemma 15** (Shen and Yang [21]). If  $\kappa \geq 7$ , then the graph  $G$  contains no configurations in Fig. 1e, 1f, 1g and 1h.

**Lemma 16** (Du et al. [5]). If  $\kappa \geq 7$ , then the graph  $G$  contains no configurations in Fig. 2a and Fig. 2b.

**Lemma 17.** If  $\kappa \geq 7$ , then the graph  $G$  contains no configuration in Fig. 2c.

**Proof.** Suppose that the edge  $vv_2$  is contained in two triangles  $vv_1v_2$  and  $vv_2v_3$ . We further assume that  $v_2$  is a 3-vertex and  $v$  is adjacent to a 2-vertex  $u$ . By the minimality of  $G$ , the graph  $G - uv$  has a total coloring with at most  $\kappa$  colors. We erase the colors on vertices  $u$  and  $v_2$ , and denote the resulting coloring by  $\varphi$ . Without loss of generality, let  $\varphi(vv_1) = 1$ ,  $\varphi(vv_2) = 2$  and  $\varphi(vv_3) = 3$ . Note that  $[\kappa]$  is the disjoint union of  $\mathcal{U}_\varphi(v)$  and  $\{\varphi(uv)\}$ , where  $w$  is the neighbor of  $u$  other than  $v$ . Without loss of generality, we assume that  $\varphi(uw) = \kappa$ . If  $\kappa \notin \{\varphi(v_1v_2), \varphi(v_2v_3)\}$ , then we recolor  $vv_2$  with  $\kappa$  and  $uv$  with 2. By symmetry, we assume that  $\varphi(v_1v_2) = \kappa$ . If  $\varphi(v_2v_3) \neq 1$ , then we recolor  $vv_1, v_1v_2$  and  $uv$  with  $\kappa, 1$  and 1, respectively. If  $\varphi(v_2v_3) = 1$ , then we recolor  $vv_1, v_1v_2, v_2v_3, vv_3$  and  $uv$  with  $\kappa, 1, 3, 1$  and 3, respectively.  $\square$

**Lemma 18.** If  $\kappa \geq 7$ , then  $G$  contains no  $(4, 4, 4)$ -triangle.

**Proof.** Suppose that  $uvw$  is a  $(4, 4, 4)$ -triangle. The graph  $G - \{uv, vw, uw\}$  admits a total coloring with at most  $\kappa$  colors. Now, we erase the colors on the vertices  $u, v$  and  $w$ , and denote the resulting coloring by  $\varphi$ . Note that each element in  $\{u, v, w, uv, vw, uw\}$  forbids at most four colors and each element has at least three available colors. Thus, we can extend  $\varphi$  to  $G$  by using the fact that every triangle is totally 3-choosable [9, Theorem 2.2].  $\square$

### 3 Total coloring of planar and toroidal graphs

McDiarmid and Sánchez-Arroyo [17] gave a general upper bound in terms of the maximum degree (the graph is not necessarily planar or toroidal).

**Theorem 3.1** ([17]). If  $G$  is a simple graph with maximum degree  $\Delta$ , then  $\chi''(G) \leq \frac{7}{5}\Delta + 3$ .

**Theorem 3.2.** Let  $G$  be a planar or toroidal graph with maximum degree at most  $\kappa - 1$ . If  $\kappa \geq 11$ , then  $\chi''(G) \leq \kappa$ .

**Proof.** Let  $G$  be a counterexample to the theorem with the minimum number of edges. Thus, it is a  $\kappa$ -deletion-minimal graph, and all the properties of  $\kappa$ -deletion-minimal graph hold for  $G$ . By Theorem 3.1, we assume that  $\Delta(G) \geq 7$ . We also assume that  $G$  has been embedded in the corresponding surface. Let  $F(G)$  denote the face set of  $G$ . By Lemma 1, the graph  $G$  is 2-connected and  $\delta(G) \geq 2$ . The degree  $\deg_G(f)$  of a face  $f$  is the number of edges with which it is incident, and every cut edge being counted twice.

**Claim 1** (Kowalik et al. [12]). Every vertex is adjacent to at most one 2-vertex.

From Euler's formula, we have the following equality:

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -8 \text{ or } 0. \quad (1)$$

Assign the initial charge of every vertex  $v$  to be  $\deg(v) - 4$  and the initial charge of every face  $f$  to be  $\deg(f) - 4$ . We design appropriate discharging rules and redistribute charges among vertices and faces, such that the final charge of every vertex and every face is nonnegative; moreover, the final charge of every vertex with maximum degree is positive, which derives a contradiction.

A 2-vertex is *good* if it is incident with a  $5^+$ -face, otherwise, it is *bad*.

**The Discharging Rules:**

- (R1) If  $f$  is a  $(4^-, 8^+, 8^+)$ -face, then  $f$  receives  $\frac{1}{2}$  from each incident  $8^+$ -vertex.
- (R2) If  $f$  is a  $(5, 7^+, 7^+)$ -face, then  $f$  receives  $\frac{1}{5}$  from the incident 5-vertex and  $\frac{2}{5}$  from each incident  $7^+$ -vertex.
- (R3) If  $f$  is a  $(6^+, 6^+, 6^+)$ -face, then  $f$  receives  $\frac{1}{3}$  from each incident vertex.
- (R4) Every vertex sends 1 to each adjacent bad 2-vertex and  $\frac{1}{2}$  to each adjacent good 2-vertex.
- (R5) Every  $5^+$ -face sends 1 to each incident 2-vertex.
- (R6) Every 3-vertex receives  $\frac{1}{3}$  from each adjacent vertex.

By Lemma 2 and the discharging rules, the final charge of every 3-face is nonnegative. Clearly, the final charge of every 4-face is zero.

By Lemma 2 and Claim 1, a  $\tau$ -face is incident with at most  $\lfloor \frac{\tau}{3} \rfloor$  vertices of degree two. If  $f$  is a  $\tau$ -face with  $\tau \geq 5$ , then the final charge is at least  $\tau - 4 - \lfloor \frac{\tau}{3} \rfloor \times 1 \geq 0$  by (R5).

**Let  $v$  be a 2-vertex.** If it is bad, then the final charge is  $2 - 4 + 2 \times 1 = 0$  by (R4). If it is good, then the final charge is at least  $2 - 4 + 1 + 2 \times \frac{1}{2} = 0$  by (R4).

**Let  $v$  be a 3-vertex.** The final charge is  $3 - 4 + 3 \times \frac{1}{3} = 0$  by (R6).

**Let  $v$  be a 4-vertex.** Clearly, the final charge is equal to the initial charge zero.

**Let  $v$  be a 5-vertex.** The final charge is at least  $5 - 4 - 5 \times \frac{1}{5} = 0$  by (R2).

**Let  $v$  be a 6-vertex.** The final charge is at least  $6 - 4 - 6 \times \frac{1}{3} = 0$  by (R3).

**Let  $v$  be a 7-vertex.** The final charge is at least  $7 - 4 - 7 \times \frac{2}{5} > 0$  by (R2) and (R3).

**Let  $v$  be an 8-vertex.** The final charge is at least  $8 - 4 - 8 \times \frac{1}{2} = 0$  by (R1), (R2) and (R3). Moreover, the final charge equals zero only if  $v$  is incident with eight 3-faces and every incident 3-face contains a 4-vertex, but this is impossible by Lemma 2 and Lemma 7. Hence, the final charge of  $v$  is positive.

**Let  $v$  be a  $t$ -vertex with  $9 \leq t \leq \kappa - 3$ .** The final charge is at least  $t - 4 - t \times \frac{1}{2} = \frac{t-8}{2} > 0$  by (R1), (R2) and (R3).

**Let  $v$  be a  $(\kappa - 2)$ -vertex.** Note that  $v$  is not adjacent to 2-vertices. If  $v$  is not adjacent to 3-vertices, then the final charge is at least  $(\kappa - 2) - 4 - (\kappa - 2) \times \frac{1}{2} = \frac{\kappa-10}{2} > 0$ . So we may assume that  $v$  is adjacent to a 3-vertex  $u$ . Suppose that  $uv$  is contained in a triangle  $uvw$ . By Lemma 7, the vertex  $v$  is adjacent to exactly one 3-vertex. Thus, the final charge is

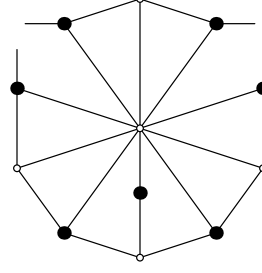


Fig. 3

at least  $(\kappa - 2) - 4 - (k - 2) \times \frac{1}{2} - \frac{1}{3} = \frac{\kappa - 10}{2} - \frac{1}{3} > 0$ . So we may further assume that every edge  $uv$  with  $u$  is a 3-vertex is not contained in a triangle. Thus, we have that the number of adjacent 3-vertices and incident 3-faces is at most  $\kappa - 2$ , and the final charge is at least  $(\kappa - 2) - 4 - (k - 3) \times \frac{1}{2} - \frac{1}{3} = \frac{\kappa - 9}{2} - \frac{1}{3} > 0$ .

**Let  $v$  be a  $(\kappa - 1)$ -vertex.**

A *fan* is a subgraph with some (at least one) consecutive 3-faces such that  $vv_0v_1, vv_1v_2, \dots, vv_{s-1}v_s$  are the boundaries and each of  $vv_0$  and  $vv_s$  is incident with a  $4^+$ -face. Let  $\epsilon$  be the number of fans at  $v$ .

(a) The vertex  $v$  is adjacent to a 2-vertex.

By Claim 1, the vertex  $v$  is adjacent to exactly one 2-vertex  $w$ . First of all, suppose that  $w$  is in a triangle. By Lemma 10, no edge  $uv$  with  $u$  is a 3-vertex is incident with 3-faces. Thus, the number of adjacent 3-vertices and incident 3-faces is at most  $\kappa - 2$ , and thus the final charge is at least  $(\kappa - 1) - 4 - 1 - (k - 2) \times \frac{1}{2} = \frac{\kappa - 10}{2} > 0$ .

Next, we may assume that the 2-vertex  $w$  is not in a triangle. By Lemma 17, each edge  $uv$  with  $u$  is a 3-vertex is contained in at most one triangle, and thus each fan contains at most two 3-vertices. Moreover, if a fan contains exactly two 3-vertices, then the fan contains at least two 3-faces by Lemma 2. Note that  $\epsilon$  is at most  $\frac{\kappa - 2}{2}$  because the 2-vertex  $w$  is incident with two  $4^+$ -faces.

Note that the number of adjacent 3-vertices and incident 3-faces is at most  $(\kappa - 2) + \epsilon$ ; the number of  $4^+$ -face is at least  $\epsilon + 1$ .

If  $\kappa \geq 12$ , then the final charge of  $v$  is at least

$$(\kappa - 1) - 4 - 1 - (\kappa - 2 - \epsilon) \times \frac{1}{2} - 2\epsilon \times \frac{1}{3} = \frac{\kappa - 10}{2} - \frac{\epsilon}{6} \geq \frac{5\kappa - 58}{12} > 0.$$

Now, we consider the case  $\kappa = 11$ . Note that  $\epsilon \leq 4$ . If  $\epsilon = 4$ , then at least three fans only contains one 3-face and each such 3-face contains at most one 3-vertex, and then the final charge of  $v$  is at least  $10 - 4 - 1 - 5 \times \frac{1}{2} - 5 \times \frac{1}{3} > 0$ . If  $\epsilon = 3$ , then the final charge of  $v$  is at least  $10 - 4 - 1 - 6 \times \frac{1}{2} - 6 \times \frac{1}{3} = 0$ . Moreover, the final charge equals zero only if the local structure is as illustrated in Fig. 3 (note that the 2-vertex  $w$  is incident with two  $4$ -faces), but it is excluded by Fig. 1d. Hence, the final charge of  $v$  is positive.

If  $\epsilon = 2$ , then the final charge is at least  $10 - 4 - 1 - 7 \times \frac{1}{2} - 4 \times \frac{1}{3} > 0$ . If  $\epsilon = 1$ , then the final charge is at least  $10 - 4 - 1 - 8 \times \frac{1}{2} - 2 \times \frac{1}{3} > 0$ . If  $\epsilon = 0$ , then the final charge is at least  $10 - 4 - 1 - 9 \times \frac{1}{3} > 0$ .

(b) The vertex  $v$  is not adjacent to 2-vertices.

If  $v$  is not incident with  $4^+$ -faces, then according to Lemma 11 and Lemma 12, the vertex  $v$  is adjacent to at most two 3-vertices, and then the final charge is at least  $(\kappa - 1) - 4 - (\kappa - 1) \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{\kappa - 9}{2} - \frac{2}{3} > 0$ . So we may assume that  $v$  is incident with at least one  $4^+$ -face.

(•) Every fan contains at most three 3-vertices by Lemma 11 and Lemma 12.

If  $\epsilon = 0$ , then the final charge is at least  $(\kappa - 1) - 4 - (\kappa - 1) \times \frac{1}{3} = \frac{2\kappa - 14}{3} > 0$ . If  $\epsilon = 1$ , then the final charge is at least  $(\kappa - 1) - 4 - (\kappa - 2) \times \frac{1}{2} - 3 \times \frac{1}{3} = \frac{\kappa - 10}{2} > 0$ . If  $\epsilon = 2$ , then according to (•), Lemma 11 and Lemma 12, the vertex  $v$  is adjacent to at most five 3-vertices on the two fans, and then the final charge is at least  $(\kappa - 1) - 4 - (\kappa - 3) \times \frac{1}{2} - 5 \times \frac{1}{3} = \frac{\kappa - 7}{2} - \frac{5}{3} > 0$ . If  $\epsilon \geq 3$  and every fan at  $v$  contains at most two 3-vertices, then the final charge of  $v$  is at least  $(\kappa - 1) - 4 - (\kappa - 1 - \epsilon) \times \frac{1}{2} - 2\epsilon \times \frac{1}{3} = \frac{\kappa - 9}{2} - \frac{\epsilon}{6} \geq \frac{\kappa - 9}{2} - \frac{\kappa - 1}{12} > 0$ . In the next,

we assume that  $\epsilon \geq 3$  and there exists a fan containing exactly three 3-vertices. By Lemma 11 and Lemma 12, all the other fans contain at most two 3-vertices, thus  $v$  is adjacent to at most five 3-vertices on fans. Hence, the final charge of  $v$  is at least  $(\kappa - 1) - 4 - (\kappa - 1 - \epsilon) \times \frac{1}{2} - 5 \times \frac{1}{3} = \frac{\kappa - 9 + \epsilon}{2} - \frac{5}{3} > 0$ .

Now, we have checked that the final charge of every vertex and every face is nonnegative. Let  $w$  be a vertex with maximum degree. Clearly, the vertex  $w$  is a  $7^+$ -vertex. From the above arguments, we have that  $w$  has positive final charge, thus the sum of the final charge of every element is positive, which leads to a contradiction.  $\square$

**Corollary 1.** If  $G$  is a planar or toroidal graph with maximum degree at least 10, then  $\chi''(G) \leq \Delta(G) + 1$ .

**Corollary 2.** If  $G$  is a planar or toroidal graph with maximum degree at least 9, then  $\chi''(G) \leq \Delta(G) + 2$ .

## References

- [1] M. Behzad, Graphs and their chromatic numbers, Ph.D. thesis, Michigan State University (1965).
- [2] O. V. Borodin, On the total coloring of planar graphs, *J. Reine Angew. Math.* 394 (1989) 180–185.
- [3] O. V. Borodin, A. V. Kostochka and D. R. Woodall, List edge and list total colourings of multigraphs, *J. Combin. Theory Ser. B* 71 (1997) (2) 184–204.
- [4] O. V. Borodin, A. V. Kostochka and D. R. Woodall, Total colorings of planar graphs with large maximum degree, *J. Graph Theory* 26 (1997) (1) 53–59.
- [5] D. Du, L. Shen and Y. Wang, Planar graphs with maximum degree 8 and without adjacent triangles are 9-totally-colorable, *Discrete Appl. Math.* 157 (2009) (13) 2778–2784.
- [6] J. Hou, G. Liu and J. Cai, List edge and list total colorings of planar graphs without 4-cycles, *Theoret. Comput. Sci.* 369 (2006) (1-3) 250–255, claims.
- [7] J. Hou, J. Wu, G. Liu and B. Liu, Total coloring of embedded graphs of maximum degree at least ten, *Sci. China Math.* 53 (2010) (8) 2127–2133.
- [8] T. R. Jensen and B. Toft, Graph coloring problems, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1995.
- [9] M. Juvan, B. Mohar and R. Škrekovski, List total colourings of graphs, *Combin. Probab. Comput.* 7 (1998) (2) 181–188.
- [10] A. V. Kostochka, The total coloring of a multigraph with maximal degree 4, *Discrete Math.* 17 (1977) (2) 161–163.
- [11] A. V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, *Discrete Math.* 162 (1996) (1-3) 199–214.
- [12] Ł. Kowalik, J.-S. Sereni and R. Škrekovski, Total-coloring of plane graphs with maximum degree nine, *SIAM J. Discrete Math.* 22 (2008) (4) 1462–1479.
- [13] R. Li and B. Xu, Edge choosability and total choosability of toroidal graphs without intersecting triangles, *Ars Combin.* 103 (2012) 109–118.
- [14] B. Liu, J. Hou and G. Liu, List edge and list total colorings of planar graphs without short cycles, *Inform. Process. Lett.* 108 (2008) (6) 347–351.
- [15] B. Liu, J. Hou, J. Wu and G. Liu, Total colorings and list total colorings of planar graphs without intersecting 4-cycles, *Discrete Math.* 309 (2009) (20) 6035–6043.
- [16] R. Luo and C.-Q. Zhang, Total chromatic number of graphs with small genus, in *The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications*, vol. 11 of *Electron. Notes Discrete Math.*, Elsevier, Amsterdam, 2002, pp. 468–477.

- [17] C. J. H. McDiarmid and A. Sánchez-Arroyo, An upper bound for total colouring of graphs, *Discrete Math.* 111 (1993) (1-3) 389–392.
- [18] M. Rosenfeld, On the total coloring of certain graphs, *Israel J. Math.* 9 (1971) (3) 396–402.
- [19] D. P. Sanders and J. Maharry, On simultaneous colorings of embedded graphs, *Discrete Math.* 224 (2000) (1-3) 207–214.
- [20] D. P. Sanders and Y. Zhao, On total 9-coloring planar graphs of maximum degree seven, *J. Graph Theory* 31 (1999) (1) 67–73.
- [21] L. Shen and Y. Wang, Total colorings of planar graphs with maximum degree at least 8, *Sci. China Ser. A* 52 (2009) (8) 1733–1742.
- [22] X.-Y. Sun, J.-L. Wu, Y.-W. Wu and J.-F. Hou, Total colorings of planar graphs without adjacent triangles, *Discrete Math.* 309 (2009) (1) 202–206.
- [23] N. Vijayaditya, On total chromatic number of a graph, *J. London Math. Soc.* (2) 3 (1971) (3) 405–408.
- [24] V. G. Vizing, Critical graphs with given chromatic class, *Metody Diskret. Analiz.* 5 (1965) 9–17.
- [25] H. Wang, B. Liu, J. Wu and G. Liu, Total coloring of embedded graphs with maximum degree at least seven, *Theoret. Comput. Sci.* 518 (2014) 1–9.
- [26] H. Wang, B. Liu, J. Wu and B. Wang, Total coloring of graphs embedded in surfaces of nonnegative Euler characteristic, *Sci. China Math.* 57 (2014) (1) 211–220.
- [27] P. Wang and J.-L. Wu, A note on total colorings of planar graphs without 4-cycles, *Discuss. Math. Graph Theory* 24 (2004) (1) 125–135.
- [28] T. Wang, Total coloring of 1-toroidal graphs of maximum degree at least 11 and no adjacent triangles, eprint arXiv:1206.3862 (2012).
- [29] W. Wang, Total chromatic number of planar graphs with maximum degree ten, *J. Graph Theory* 54 (2007) (2) 91–102.
- [30] J. Wu and P. Wang, List-edge and list-total colorings of graphs embedded on hyperbolic surfaces, *Discrete Math.* 308 (2008) (24) 6210–6215.
- [31] H. P. Yap, Total colourings of graphs, *Bull. London Math. Soc.* 21 (1989) (2) 159–163.
- [32] Y. Zhao, On the total coloring of graphs embeddable in surfaces, *J. London Math. Soc.* (2) 60 (1999) (2) 333–343.